Journal of Engineering Mathematics **26**: 297–313, 1992.

On the development of Görtler vortices in wall jet flow

P.D. WADEY

Department of Mathematics, North Park Road, University of Exeter, Devon EX4 4QE, England; present address: British Aerospace (Space Systems) Ltd., Argyle Way, Stevenage, Hertfordshire, SG1 2AS, England

Received 5 September 1990; accepted in revised form 11 February 1991

Abstract. The development of Görtler vortices in wall jet flow over curved surfaces is considered in both the linear and nonlinear growth régimes. It is shown, using asymptotic methods based on the largeness of the wavenumber of the vortices, that this hydrodynamic instability is prone to occur more readily on concave rather than convex surfaces. It is found that after passing the position of neutral stability, the flow develops a surprising structure quite unlike that produced in the Blasius boundary-layer. Once the flow is into the unstable regime, the effect of increasing the Görtler number is to move the vortices away from the wall.

§1. Introduction

We shall consider some stability characteristics of Görtler vortices in wall-jet flow over a curved surface. Most previous investigations have been limited to the study of this centrifugal instability in a Blasius boundary layer, where the monotonicity of the velocity profile requires that the surface be concave in order for the instability to become manifest. With the wall-jet velocity profile, however, flow over plates of either curvature can produce vortices. Glauert [1] gave the solution to the boundary-layer equations for the wall jet, which is found when, for example, a jet of air impinges on a surface. Carpenter et al. [2] demonstrated experimentally how Görtler vortices may be found over Coanda surfaces. Such surfaces have industrial applications in waste-gas flares.

Görtler [3] considered the stability of incompressible boundary layers over a slightly curved surface, and showed how a secondary flow consisting of vortices aligned with the principal direction of motion can be supported. His equations were solved approximately and produced the result that boundary layers on concave curved walls are unstable at sufficiently high flow speeds. The stability of the flow depends on the Görtler parameter, G, which is related to the Reynolds number of the flow and the local concave or convex curvature of the wall.

Hämmerlin [4] solved Görtler's equations exactly and found the physically unacceptable result that the critical Görtler number, G_C, below which no instability could be found, corresponded to vortices of infinite wavelength. By retaining some higher order curvature terms, he re-derived the equations [5], the solution of which yielded a critical Görtler number at non-zero wavenumber.

Numerous authors have considered modifications to Hämmerlin's equations. Herbert [6] and Floryan and Saric [7] gave reviews of previous work, and the latter authors presented results concerning the effect of suction on the vortex mechanism. Floryan [8] has discussed the stability of the wall jet to Görtler vortices, mentioning the result that this velocity field admits the instability on convex walls.

However, all the above authors used parallel-flow approximations, and their results disagree for vortices with small wavenumbers. Here, by 'parallel-flow', we mean that some of the terms in the governing equations have been neglected or replaced by simplified terms in a manner which does not reflect the structure of the Görtler problem correctly. For example, Floryan [8] assumes that all disturbance quantities vary as $\exp(\beta x)$ where x is the downstream co-ordinate, and β is the downstream spatial growth rate. Hall [9] showed that the parallel-flow theories had no mathematical justification except for high wavenumbers. He developed a formal asymptotic expansion of the appropriate linear stability equations based on the smallness of the wavelength of the imposed disturbance, and showed that the position for neutral stability in this high wavenumber régime depends on the form and location of the initial disturbance from the mean flow. It was shown that the vortices form in a region of thickness $O(\varepsilon^{1/2})$, where $2\pi\varepsilon$ is the (small) wavelength of the vortices under consideration, centred on the point where $\bar{u}\bar{u}_y$ is a local maximum, that is, the point at which Rayleigh's [10] stability criterion is most violated.

Stuart [11] and Watson [12] showed how non-linear effects could be taken into account close to the conditions of neutral stability under linear theory for plane poiseuille flow and for Couette flow. However, their approach required the correction to the mean flow to be an order of magnitude smaller than the mean flow itself. Hall [13] has shown that this is not the case for the Görtler problem. He considered the weakly non-linear development of a locally neutrally stable vortex. He showed how in an ε-neighbourhood of the point of neutral stability according to linear theory, X_n say, the downstream velocity component due to the vortex is of the same magnitude as that of the mean flow. At an asymptotically large distance downstream of X_n, it was shown how the 'correction' to the mean flow becomes an order of magnitude larger than the mean flow itself, for the Blasius boundary layer. At this stage, the vortex flow extends beyond the boundary layer into the free stream.

The full, linear, partial-differential equations were solved numerically by Hall [14] for vortices of $O(1)$ wavenumber. An initial disturbance was imposed on the flow at some location and its downstream development was examined. The growth rate of the disturbance, based on a non-dimensional energy function associated with the flow, was found to depend on the form of the initial disturbance and the place at which it had been introduced to the flow. Hall concluded that no unique neutral curve for the Görtler instability exists. He did, however, show that the different neutral curves merged to form one curve in the high wavenumber regime and that this was the curve which parallel-flow theorists had produced. This confirms the idea that such theories are only valid in this regime, and are of little use elsewhere in connection with the Görtler problem.

Hall and Lakin [15] gave an account of the fully non-linear problem, using Hall [13] as a starting point. They showed how the region of vortex activity increases to an $O(1)$ depth. The way in which the vortex disturbance in the core decays away in two bounding shear layers was given, and asymptotic solutions for the initial and ultimate forms of the instability were calculated. At an $O(1)$ distance from the point where the vortices start to grow, a numerical calculation was performed in order to find the upper and lower bounds for the region of disturbance activity. The mean flow was found to be so changed by the presence of the vortex that it bears no relationship to the unperturbed state; indeed, the vortex was shown to drive the mean flow.

We shall use the approach of Hall [9] to show how the right-hand branch of the neutral curve may be generated for wall-jet flow. We show how our results compare with those of Floryan [8] who used a normal mode approach. It is found that the third term in the asymptotic expansion of the neutral curve depends on the position in the flow where the requirement for neutral stability is applied. We develop the non-linear stability theory for

Görtler vortices along the lines of Hall and Lakin [15], and show how the instability initially develops from its position of neutral stability. It is found that there is an interesting asymptotic structure in which the Görtler number can be scaled out of the problem, and that this leads to a different structure for the flow after the onset of vortices from that which pertains in the Blasius boundary-layer situation.

The remainder of this paper is set out as follows: in §2, we give the formulation of the problem; in §3 we consider the solution of the linearised disturbance equations for asymptotically high wavenumber solutions, after the method of Hall [9], and in §4 the non-linear problem is treated, in the manner of Hall and Lakin [15]. Some final remarks on the problem are given in §5.

§2. Formulation of the problem

We consider an incompressible steady fluid flow of kinematic viscosity ν and density ρ over a section of a curved wall having a curvature $R^{-1}K(x^*/l)$. We take l to be a typical length measured along the wall and U_0 be a typical downstream velocity. x^* is measured along the wall, y^* perpendicular to the wall, and z^* so that (x^*, y^*, z^*) form an orthogonal triad, where * denotes a dimensional quantity. The corresponding velocity components are (u^*, v^*, w^*). We define the Reynolds number $\mathrm{Re} = U_0 l/\nu$, and a curvature parameter $\delta = l/R$. We shall confine our attention to the double limit $\mathrm{Re}\to\infty$, $\delta\to 0$, with the Görtler number $G = 2\,\mathrm{Re}^{1/2}\delta$ held fixed at $O(1)$. We non-dimensionalise via

$$(x, y, z) = l^{-1}(x^*, y^*\,\mathrm{Re}^{-1/2}, z^*\,\mathrm{Re}^{-1/2}),$$
$$(u^*, v^*, w^*) = U_0(u, \mathrm{Re}^{-1/2}v, \mathrm{Re}^{-1/2}w).$$

Here, we are expecting the spanwise dependence of the flow to occur on the boundary-layer length-scale. The Navier–Stokes equations for the flow may then be written in the form

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,\\
&\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},\\
&\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} - \frac{1}{2}GKu^2 - \frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z},\\
&\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} - \frac{\partial p}{\partial z} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z},
\end{aligned} \tag{2.1a–d}$$

where terms of relative order $\mathrm{Re}^{-1/2}$ have been neglected, and the pressure has been scaled on $\rho U_0^2\,\mathrm{Re}^{-1}$. In the absence of any Görtler vortices, we can write $(u, v, w) = (\bar{u}, \bar{v}, 0)$ where $\bar{u}$ and $\bar{v}$ are the velocity components of the wall jet velocity field given by Glauert [1] as

$$\begin{aligned}
\bar{u} &= \tfrac{1}{4}x^{-1/2}f'(\xi),\\
\bar{v} &= \tfrac{1}{4}x^{-3/4}(3\xi f'(\xi) - f(\xi)),
\end{aligned} \tag{2.2}$$

again, correct to $O(\mathrm{Re}^{-1/2})$. Here $\xi = \frac{1}{4} y x^{-3/4}$ and f satisfies the ordinary differential equation

$$f''' + ff'' + 2f'^2 = 0 , \tag{2.3a}$$

subject to the boundary conditions

$$f(0) = f'(0) = 0 , \qquad f'(\infty) = 0 . \tag{2.3b}$$

We now perturb the basic velocity and pressure fields to obtain

$$(u^*, v^*, w^*) = U_0(\bar{u} + U, \mathrm{Re}^{-1/2}(\bar{v} + V), \mathrm{Re}^{-1/2} W) ,$$
$$p^* = \frac{\rho U_0^2}{\mathrm{Re}} (\bar{p} + P) .$$

When the altered flow-field expressions are substituted into the equations of motion, the resulting equations linearised and the mean flow equations subtracted from them, we have

$$\begin{aligned} &U_x + V_y + W_z = 0 , \\ &U_{yy} + U_{zz} - V\bar{u}_y = \bar{u}U_x + \bar{u}_x U + \bar{v}U_y , \\ &V_{yy} + V_{zz} - KG\bar{u}U - P_y = \bar{u}V_x + U\bar{v}_x + \bar{v}V_y + V\bar{v}_y \\ &W_{yy} + W_{zz} - P_z = \bar{u}W_x + \bar{v}W_y . \end{aligned} \tag{2.4a–d}$$

These equations will be used as the starting point for our linear asymptotic calculations at high wavenumber, whereas the non-linear theory will be based on equations (2.1).

§3. Linear theory

Here, we show how the neutral curve in the high wavenumber régime may be determined. Following the method of Hall [9], suppose now that U, V, W and P are each taken to be proportional to $\exp(iz/\varepsilon)$, where ε^{-1}, the non-dimensional wavenumber of the disturbance, is large. Then, $\partial/\partial z \sim 1/\varepsilon$ and so the x- and y-momentum equations yield the balances $U = O(\varepsilon^2 V)$ and $V = O(\varepsilon^2 GU)$ respectively. For consistency, we must have $G = O(\varepsilon^{-4})$. Hall showed how small-wavelength Görtler vortices position themselves in a layer of thickness $\varepsilon^{1/2}$ centred where $\bar{u}\bar{u}_y$ is a maximum. For wall-jet flow this maximum is at $\xi = \xi^+ = 1.1$ for the concave wall. In addition, unlike in the case of the Blasius boundary layer, the wall jet is also unstable where $|\bar{u}\bar{u}_y|$ is a maximum on the convex wall. This occurs for $\xi = \xi^+ = 2.9$. Thus, $\partial/\partial y \sim \varepsilon^{-1/2}$ and we obtain the scalings for W and P from the continuity and z-momentum equations respectively as $W = O(\varepsilon^{1/2} V)$ and $P = O(\varepsilon^{-1/2} V)$. We define the variable η by

$$\eta = \varepsilon^{-1/2}(\xi - \xi^+) ,$$

and so equations (2.4) must be rewritten with $\partial/\partial y$ replaced by $\frac{1}{4} x^{-3/4} \varepsilon^{-1/2}\, \partial/\partial\eta$, and $\partial/\partial x$ by

$$\frac{\partial}{\partial x} - \frac{3}{4}\xi x^{-1}\varepsilon^{-1/2}\frac{\partial}{\partial \eta}.$$

The disturbance is on a length scale of $O(\varepsilon)$ in the z-direction. The y-momentum equation suggests that the growth of any Görtler vortex will be on a length scale such that $\partial^2/\partial z^2 \sim \partial/\partial x \sim \varepsilon^{-2}$. We therefore assume that U, V, W, and P are each proportional to E, where

$$E = \exp\left(\frac{\mathrm{i}z}{\varepsilon} + \frac{1}{\varepsilon^2}\int^x (\beta_0(\phi) + \varepsilon^{1/2}\beta_1(\phi) + \cdots)\,\mathrm{d}\phi\right), \tag{3.1}$$

and the growth rate $\beta_0 + \varepsilon^{1/2}\beta_1 + \cdots$ is to be calculated. On the basis of the above discussion, we expand our disturbance quantities as

$$\begin{aligned}
U &= \varepsilon^2(U_0 + \varepsilon^{1/2}U_1 + \varepsilon U_2 + \cdots)E + \text{C.C.}\,, \\
V &= (V_0 + \varepsilon^{1/2}V_1 + \varepsilon V_2 + \cdots)E + \text{C.C.}\,, \\
W &= \varepsilon^{1/2}(W_0 + \varepsilon^{1/2}W_1 + \varepsilon W_2 + \cdots)E + \text{C.C.}\,, \\
P &= \varepsilon^{-1/2}(P_0 + \varepsilon^{1/2}P_1 + \varepsilon P_2 + \cdots)E + \text{C.C.}\,,
\end{aligned} \tag{3.2}$$

where 'C.C.' denotes 'complex conjugate', and replace G by the expansion

$$G = \varepsilon^{-4}(g_0 + \varepsilon^{1/2}g_1 + \varepsilon g_2 + \cdots).$$

We take U evaluated at $\xi_1 = \frac{1}{4}y_1x_1^{-3/4}$ to be a representative disturbance quantity and consider its relative change as the flow moves downstream:

$$\frac{1}{U}\frac{\partial U}{\partial x}\bigg|_{\xi=\xi_1} = \frac{\beta_0}{\varepsilon^2} + \frac{\beta_1}{\varepsilon^{3/2}} + \frac{\beta_2}{\varepsilon} + \frac{1}{\varepsilon^{1/2}}\left(\beta_3 - \frac{3\xi^+}{4x}\frac{\partial U_0}{\partial \eta}\bigg|_{\xi=\xi_1}\right) + O(\varepsilon^0).$$

For zero growth at x_1, y_1, we require $U^{-1}\,\partial U/\partial x = 0$, so that to $O(\varepsilon^{-1/2})$ we choose

$$\beta_0 = \beta_1 = \beta_2 = 0 \quad\text{and}\quad \beta_3 = \frac{3\xi^+}{4x}\frac{\partial U_0}{\partial \eta}\frac{1}{U_0}\bigg|_{\xi=\xi_1}. \tag{3.3}$$

We also need expansions for the mean flow quantities $\bar{u}$ and $\bar{v}$ defined by the Taylor expansions about (x^+, y^+) as

$$\begin{aligned}
\bar{u} &= \tfrac{1}{4}x^{-1/2}(u_0^+ + \eta\varepsilon^{1/2}u_1^+ + \eta^2\varepsilon u_2^+ + \cdots) \quad\text{and} \\
\bar{v} &= \tfrac{1}{4}x^{-3/4}(v_0^+ + \eta\varepsilon^{1/2}v_1^+ + \eta^2\varepsilon v_2^+ + \cdots),
\end{aligned} \tag{3.4}$$

with

$$u_k^+ = \frac{1}{k!}\frac{\partial^k(f')}{\partial \xi^k}\bigg|_{\xi=\xi^+} \quad\text{and}\quad v_k^+ = \frac{1}{k!}\frac{\partial^k(3\xi f - f')}{\partial \xi^k}\bigg|_{\xi=\xi^+}.$$

We substitute the expansions (3.2) and (3.4) into the governing equations, (2.4), and equate coefficients of like powers of ε. Consistency of the leading-order terms in the momentum equations (2.4b, c) requires

$$Kg_0 = \frac{64x^{7/4}}{u_0^+ u_1^+}\,, \tag{3.5a}$$

whilst the following expressions for U_0, W_0, and P_0 are found from equations (2.4a–d) in terms of V_0 which is determined at higher order:

$$\begin{aligned} U_0 &= -\tfrac{1}{16}x^{-5/4}u_1^+ V_0\,, \\ W_0 &= -\frac{\mathrm{i}}{4}\,x^{-3/4}V_{0_\eta}\,, \\ P_0 &= \tfrac{1}{4}x^{-3/4}V_{0_\eta}\,. \end{aligned} \tag{3.5b–d}$$

Consistency of the next order terms in (2.4b, c) requires

$$g_1 u_0^+ u_1^+ + \eta g_0(u_1^{+2} + 2u_0^+ u_2^+) = 0\,. \tag{3.6a}$$

However, since ξ^+ was chosen such that $\bar{u}\bar{u}_y$ is a local maximum, it is easily shown that the coefficient of ηg_0 is identically zero, and hence $g_1 = 0$. From equations (2.4a–d), the eigenfunctions in terms of V_1 are then

$$\begin{aligned} U_1 &= -\tfrac{1}{16}x^{-5/4}(u_1^+ V_1 + 2\eta u_2^+ V_0)\,, \\ W_1 &= -\frac{\mathrm{i}}{4}\,x^{-3/4}V_{1_\eta}\,, \\ P_1 &= \tfrac{1}{4}x^{-3/4}V_{1_\eta}\,, \end{aligned} \tag{3.6b–d}$$

with V_1 determined at higher order.

At the third order we shall determine the form of the zeroth-order disturbance, V_0. The x- and y-momentum equations yield at $O(E\varepsilon)$ and $O(E\varepsilon^{-1})$:

$$\begin{aligned} &\tfrac{1}{16}x^{-3/2}U_{0_{\eta\eta}} - U_2 - \tfrac{1}{16}x^{-5/4}(u_1^+ V_2 + 2\eta u_2^+ V_1 + 3\eta^2 u_3^+ V_0) = 0\,, \\ &\tfrac{1}{16}x^{-3/2}V_{0_{\eta\eta}} - V_2 - \tfrac{1}{4}x^{-3/4}P_{0_\eta} - \tfrac{1}{4}x^{-1/2}(Kg_2 u_0^+ U_0 + Kg_0(u_0^+ U_2 + \eta u_1^+ U_1 + \eta^2 u_2^+ U_0)) = 0\,. \end{aligned} \tag{3.7a, b}$$

Eliminating U_2 and V_2 simultaneously from these equations gives a solvability condition on the first-order eigenfunctions:

$$\begin{aligned} &\tfrac{1}{16}x^{-3/2}V_{0_{\eta\eta}} - \tfrac{1}{4}x^{-3/4}P_{0_\eta} + \tfrac{1}{4}x^{-1/2}Kg_0(\eta u_1^+ U_1 + \eta^2 u_2^+ U_0) - \tfrac{1}{4}x^{-1/2}Kg_2 u_0^+ U_0 \\ &\quad + \tfrac{1}{64}x^{-2}U_{0_{\eta\eta}}Kg_0 u_0^+ - \tfrac{1}{64}x^{-7/4}Kg_0 u_0^+(3V_0 u_3^+ \eta\eta + 2V_1 u_2^+ \eta) = 0\,. \end{aligned} \tag{3.8}$$

Using the expressions for P_0, U_0, V_0, and Kg_0 from (3.5), we find that (3.8) becomes

$$\frac{3}{16}\,x^{-3/2}V_{0_{\eta\eta}} + 3\eta^2 V_0\left(\frac{u_2^+}{u_0^+} + \frac{u_3^+}{u_1^+}\right) + \frac{1}{64}\,V_0 Kg_2 x^{-7/4}u_0^+ u_1^+ = 0\,. \tag{3.9}$$

Now, since ξ^+ was chosen such that $\bar{u}\bar{u}_y$ has its maximum value there,

$$\frac{u_2^+}{u_0^+} + \frac{u_3^+}{u_1^+} = -A < 0 \,.$$

We make the transformation $\theta = \eta x^{3/8}(64A)^{1/4}$ in (3.9) to obtain the familiar parabolic cylinder equation

$$\frac{\partial^2 V_0}{\partial \theta^2} - \frac{\theta^2 V_0}{4} - aV_0 = 0 \,, \tag{3.10}$$

where

$$a = -\frac{Kg_2 u_0^+ u_1^+}{96 A^{1/2} x} \,.$$

For the disturbance to be confined to the boundary layer, we need $a = -(m + \frac{1}{2})$ where m is a non-negative integer. The 'most dangerous' mode, i.e. the one for which g_{2_m} is minimised, producing the lowest neutral curve, occurs for $m = 0$. We therefore have

$$Kg_{2_0} = \frac{48 A^{1/2} x}{u_0^+ u_1^+} \,. \tag{3.11}$$

The general solution to (3.10) is $V_0 = B_1 U(a, \theta) + B_2 V(a, \theta)$, where B_1 and B_2 are arbitrary constants and U and V are the standard Parabolic Cylinder functions. In order to obtain the correct behaviour, i.e. $V_{0_m} \to 0$, as $\theta \to \pm\infty$, we choose $B_2 = 0$. We now write V_0 in terms of the Hermite polynomial of order m, He_m. Using the relationships given in Abramowitz and Stegun [16], we have, except for a normalisation constant,

$$V_{0_m} = \exp\left(-\frac{\theta^2}{4}\right) \mathrm{He}_m(\theta) \,. \tag{3.12}$$

The higher modes correspond to an increase in the number of vortices present.

At the fourth order, the first non-zero downstream-growth-rate term, β_3, is found to introduce non-parallel effects into the Görtler problem. By equating the coefficients of $O(E\varepsilon^{3/2})$ and $O(E\varepsilon^{-1/2})$ to zero in the x- and y-momentum equations, eliminating U_3 and V_3 simultaneously, and replacing lower-order quantities by the expressions in terms of V_0 and V_{0_η} via (3.5) and (3.6), we find that

$$\begin{aligned}
&\frac{3}{16} x^{-3/2} V_{1_{\eta\eta}} + 3\eta^2 V_1 \left(\frac{u_2^+}{u_0^+} + \frac{u_3^+}{u_1^+}\right) + \frac{1}{64} V_1 K g_2 x^{-7/4} u_0^+ u_1^+ \\
&= \frac{u_0^+ x^{-1/2}}{16} V_0 \left(\beta_3 - \frac{Kg_3 u_1^+ x^{-5/4}}{4}\right) - V_{0_\eta} \frac{x^{-3/2}}{8} \left((3u_0^+ \xi^+ - v_0^+) - \frac{2u_0^+}{u_1^+}\right) \\
&\quad - V_0 \left(4\eta^3 \frac{u_4^+}{u_1^+} + \frac{2\eta^3}{u_0^+ u_1^+} (2u_3^+ u_1^+ + u_2^{+2}) + \frac{\eta x^{-7/4}}{64} (u_1^{+2} K g_2 + 2u_2^+ u_0^+)\right) .
\end{aligned} \tag{3.13}$$

The left-hand side of this equation has the same form as that of (3.9), and so we make the same transformation as in the third-order problem. Thus we have,

$$\frac{\partial^2 V_1}{\partial\theta^2} - \frac{\theta^2 V_1}{4} - aV_1 = \frac{2x^{3/4}}{3A^{1/2}}\left(\frac{x^{-1/2}u_0^+ V_0}{2}\left(\beta_3 - \frac{u_1^+ Kg_3 x^{-5/4}}{32}\right) + V_0 F_1(\theta) + \frac{\partial V_0}{\partial\theta} F_2\right) \quad (3.14)$$

Here, $F_1(\theta)$ is an odd function of θ and F_2 is even in θ. Now, $\mathrm{He}_m(\theta)$ is odd if m is even and even if m is odd, so $V_0F_1 + \partial V_0/\partial\theta\, F_2$ is odd for all m. If we multiply (3.14) by $\exp(-\theta^2/4)\,\mathrm{He}_m(\theta)$ and integrate over the range $(-\infty, \infty)$, we have, after simplification,

$$\frac{x^{-1/2}u_0^+}{2}\left(\beta_3 - \frac{u_1^+ Kg_3 x^{-5/4}}{32}\right)\int_{-\infty}^{\infty}\left(V_0(\theta)\right)^2 \mathrm{d}\theta + \int_{-\infty}^{\infty} V_0\left(V_0F_1 + \frac{\partial V_0}{\partial\theta} F_2\right)\mathrm{d}\theta = 0\,.$$

The first of these integrals is positive definite. The second integral, being that of an odd function over an interval which is symmetrical about the origin, is identically zero. Thus, to satisfy the equation, we require

$$\beta_3 = \frac{u_1^+ Kg_3 x^{-5/4}}{32}\,. \quad (3.15)$$

Using the relation (3.3) for β_3, we find that we must have

$$Kg_3 = \frac{24\xi^+ x^{1/4}}{u_1^+ U_0}\left.\frac{\partial U_0}{\partial\eta}\right|_{\xi=\xi_1}. \quad (3.16)$$

That is, $g_3 = g_3(x_1, y_1)$ and so we see that g_3 is the first term in the expansion of G to depend on the position in the flow where we apply our stability criterion. Noting from (3.5b) that

$$\frac{1}{U_0}\frac{\partial U_0}{\partial\eta} = \frac{1}{V_0}\frac{\partial V_0}{\partial\eta} = \frac{V_{0_\theta}}{V_0}\,,$$

and since $\mathrm{He}_0(\theta) = 1$, we have $V_{0_\theta}/V_0 = -\theta/2$. The resulting expression for the third non-zero term in the neutral-Görtler-number expansion is then

$$Kg_3 = -\frac{12\xi^+\eta_1 x^{5/8}}{u_1^+(64A)^{1/4}}\,, \quad (3.17)$$

where η_1 is the value of η at (x_1, y_1). By noting that the $\bar{u}$ basic velocity is given by $\bar{u} = \frac{1}{4}x^{-1/2}f'(\xi)$ and that the 'real length' is xl, we are able to find the local Görtler number and wavenumber,

$$G_L = \tfrac{1}{2}x^{5/4}GK^{-1} \quad \text{and} \quad \varepsilon_L = x^{-3/4}\varepsilon\,. \quad (3.18)$$

The form of the neutral Görtler number in terms of local variables for the most dangerous mode is then

$$G_L = \varepsilon_L^{-4}\left(\frac{32}{u_0^+u_1^+} + \varepsilon_L\frac{24A^{1/2}}{u_0^+u_1^+} + \varepsilon_L^{3/2}\frac{6\xi^+\eta_1}{u_1^+(64A)^{1/4}} + O(\varepsilon_L^2)\right). \quad (3.19)$$

On substituting the numerical values of the constants (which are given in Wadey [17]) into (3.19), we find that the equations of the neutral curves for the Görtler instability in wall jet flows on concave and convex walls are

$$G_L = \varepsilon_L^{-4}(811.59 + 410.09\varepsilon_L + 16.55\varepsilon_L^{3/2}\eta_1 + O(\varepsilon_L^2)) \quad \text{and}$$
$$G_L = -\varepsilon_L^{-4}(1078.74 + 490.02\varepsilon_L + 67.85\varepsilon_L^{3/2}\eta_1 + O(\varepsilon_L^2)) , \tag{3.20}$$

respectively, where the minus sign indicates the negative curvature associated with a convex wall. At a given wavenumber, a higher Görtler number is needed to create conditions suitable for the growth of the instability on the convex wall than on the concave wall. This is consistent with the conclusions of Floryan [8] in the high wavenumber regime. Furthermore, depending on the value of η_1, the non-parallelism may be exhibited at $O(\varepsilon_L^2)$ when $\eta_1 = 0$ or $O(\varepsilon_L^{3/2})$ if η_1 is $O(1)$. The procedure becomes formally invalid if η_1 becomes $O(\varepsilon_L^{-1/2})$. As it is impossible to determine exactly the value of η_1 in experimental conditions, direct comparison of experimental and theoretical results is rendered impossible. It is this variation in η_1 which demonstrates why so-called 'parallel flow' theories are inadequate, and why there has been a large spread in experimental results for the Blasius boundary layer. The reader is referred to Hall [9] for details of these. The author is unaware of any experimental results for the laminar wall jet.

In Fig. 1 we show how the parallel flow results of Floryan [8] compare with our results. Here, G_F and ε_F are the Görtler number and wavenumber defined by Floryan; in terms of our quantities, they are given by $G_F^2 = \sqrt{0.315}G_N/2$ and $\varepsilon_F = \sqrt{0.315}\varepsilon$. The curves shown here refer to flow over a concave wall. Although Floryan's method might be expected to produce the correct neutral curve in the high wavenumber limit, in fact it appears as though only the slope of the neutral curve has been predicted correctly. In Fig. 2, we demonstrate how the variation of η_1 changes the neutral curve in the concave wall case. Similar results, shifted to the left slightly, are obtained for the convex wall. It should be remembered that these results are only formally valid in the high wavenumber regime. Despite this restriction, it may be seen that any attempt at a solution which does not allow for such variation cannot

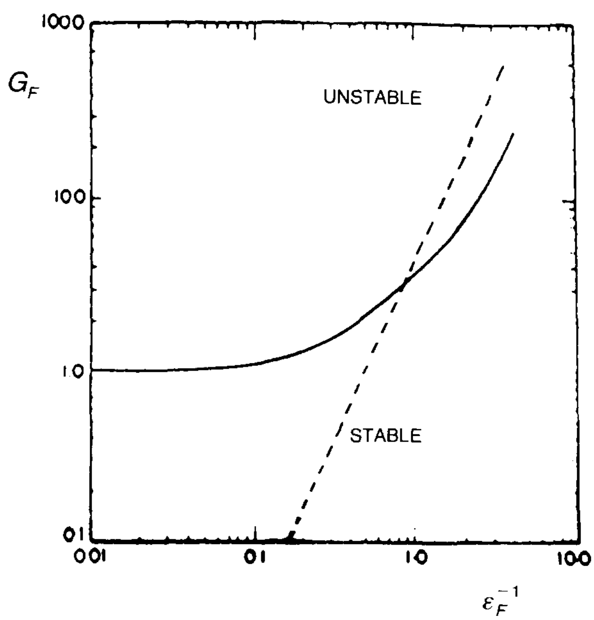

Fig. 1. Comparison of results with Floryan's [8] for the concave wall. Solid line: Floryan. Broken line: present results.

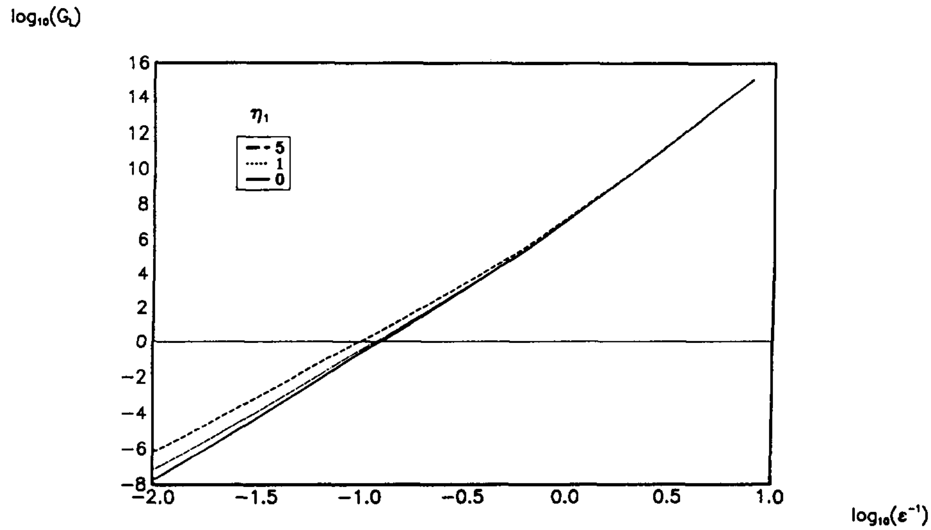

Fig. 2. The variation in neutral curves for the concave wall with η_1.

be mathematically justified, and a numerical solution such as that performed by Hall [14] is necessary to determine regions of instability at $O(1)$ wavenumbers.

§4. Non-linear theory

For the non-linear theory, we return to equations (2.1). It is helpful first to describe the manner in which Hall and Lakin [15], hereinafter referred to as HL, divided the flow into various regions downstream of its neutral stability position. In regions I and III shown in Fig. 3, there is no disturbance to the flow which would be there in the absence of a vortex. Region II, bounded above and below at ξ_U and ξ_L respectively, contains the vortices. HL showed how there exists a pair of thin shear layers of thickness $O(\varepsilon^{2/3})$ located at ξ_U and ξ_L

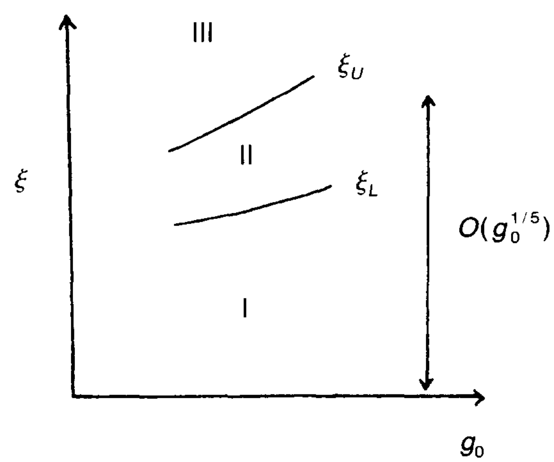

Fig. 3. The various regimes downstream of the position of neutral stability.

where the vortex activity is brought to zero, but that in these layers, the leading-order terms for the mean-flow quantities remains constant. Thus, for our purposes, we can replace these layers by simply requiring continuity of mean-flow variables at ξ_U and ξ_L.

In using the procedure developed by HL in perturbing the mean flow, the scalings are chosen such that the disturbance quantities interact with the mean flow at leading order. The same balances between terms are taken as in the linear theory, namely, between convection and diffusion terms in the x-momentum equation giving $V\bar{u}_y \sim U_{zz}$, and between diffusion and the centrifugal term in the y-momentum equation giving $KG\bar{u}U \sim V_{zz}$. The pressure gradient, P_z balances with diffusion, W_{zz} in the z-momentum equation, and from continuity we have $V_y \sim W_z$. For the non-linear theory, however, we have to take y variation to be on an $O(1)$ lengthscale. We shall now use the notation where $\bar{u}$, $\bar{v}$, and $\bar{p}$ are the parts of the flow independent of z and our expansions for the velocity components and pressure are:

$$\begin{aligned} u &= \bar{u} + (\varepsilon U_0 E + \cdots + \text{C.C.})\,, \\ v &= \bar{v} + (\varepsilon^{-1} V_0 E + \cdots + \text{C.C.})\,, \\ w &= W_0 E + \cdots + \text{C.C.}\,, \\ p &= \varepsilon^n \bar{p} + (\varepsilon^{-1} P_0 E + \cdots + \text{C.C.})\,, \end{aligned} \tag{4.1}$$

where n will be determined at leading order, and E was defined by (3.1). We now substitute the expansions into the equations of motion, (2.1). In the continuity equation, we take the terms of $O(\varepsilon^{-1}E)$, giving

$$V_{0_y} + \mathrm{i}W_0 = 0\,. \tag{4.2}$$

In the streamwise equation, (2.1b), we take the terms independent of E and ε, and terms of $O(E\varepsilon^{-1})$, giving

$$\bar{u}\bar{u}_x + \bar{v}\bar{u}_y - \bar{u}_{yy} = -V_0\tilde{U}_{0_y} - \tilde{V}_0 U_{0_y} - \mathrm{i}(\tilde{W}_0 U_0 - W_0\tilde{U}_0)\,, \tag{4.3}$$

and

$$-U_0 = V_0\bar{u}_y\,. \tag{4.4}$$

In the above, a tilde over a disturbance quantity denotes its complex conjugate. The y-momentum equation yields

$$V_0 = -Kg_0\bar{u}U_0\,, \tag{4.5}$$

for terms of $O(E\varepsilon^{-3})$, and the mean pressure is balanced by choosing $n = -4$ in the expansions, so that $\bar{p}_y = g_0K\bar{u}^2/2$. Re-writing (4.3) by replacing W_0 and U_0 with terms in V_0 via (4.2) and (4.4) yields

$$\bar{u}\bar{u}_x + \bar{v}\bar{u}_y - \bar{u}_{yy} = 2\frac{\partial}{\partial y}(\bar{u}_y|V_0|^2)\,, \tag{4.6}$$

as found in HL, and consistency of (4.3) and (4.4) requires that

$$\bar{u}\bar{u}_y = \frac{1}{Kg} . \tag{4.7}$$

This integrates to give

$$\bar{u} = q^{1/2} , \tag{4.8}$$

where we introduced for notational convenience the quantity $q = (2y + h(x))/kg_0$, with $h(x)$ an arbitrary function of x. The continuity equation provides an expression for $\bar{v}$, viz.

$$\bar{v} = \tfrac{1}{6}q^{3/2}K'g_0 - \tfrac{1}{2}h'q^{1/2} + l(x) , \tag{4.9}$$

and the integration of (4.6) yields

$$m(x) - 2\bar{u}_y|V_0|^2 = q^{-1/2}/Kg_0 + \tfrac{1}{12}K'g_0q^2 - l(x)q^{1/2} , \tag{4.10a}$$

with $m(x)$ and $l(x)$ arbitrary functions of integration, and a dash indicates a derivative with respect to x. At the boundaries of the core, $y = y_1$ and $y = y_2$, $|V_0|^2 = 0$. Eliminating $m(x)$ from equation (4.10a) evaluated at y_1 and y_2 yields the condition

$$[q^{-1/2}/Kg_0 + \tfrac{1}{12}K'g_0q^2 - l(x)q^{1/2}]_{y_1}^{y_2} = 0 . \tag{4.10b}$$

In regions I and III, the flow is independent of z and the flow quantities u and v are given by the solution to the boundary-layer equations,

$$\begin{aligned} uu_x + vu_y &= u_{yy} \\ u_x + v_y &= 0 . \end{aligned} \tag{4.11a, b}$$

These must be solved in $[0, y_1]$ and $[y_2, \infty]$ subject to the boundary conditions

$$\begin{aligned} &u = v = 0 \quad \text{at } y = 0 , \\ &u(x, y) = q^{1/2} \quad \text{at } y = y_1, y_2 , \\ &v(x, y) = \tfrac{1}{6}q^{3/2}K'g_0 - \tfrac{1}{2}h'q^{1/2} + l(x) \quad \text{at } y = y_1, y_2 \\ &\text{and} \quad u, v \to 0 \quad \text{as } y \to \infty , \end{aligned} \tag{4.11c}$$

together with condition (4.10b).

This system of equations constitutes a free-boundary problem for the core boundary positions, y_1 and y_2, and the unknown functions of integration, $h(x)$ and $l(x)$. It is doubtful that any analytic solution could be found and it would therefore be necessary to compute the solution to this partial-differential system. We are able to make some progress with the special case where the curvature is such that a similarity solution may be used, as HL did for the Blasius boundary layer. The similarity solution case is unlikely to occur in engineering situations, but the method of solution is the first step in understanding how to compute the solution to the full problem.

We now seek special forms of $h(x)$, $l(x)$, and $m(x)$ such that (4.7) and (4.10) allow the similarity type of solution for the wall jet. Comparing (4.8) with the wall-jet $\bar{u}$ velocity expression (2.2) suggests that we should take $h(x) = Hx^{-3/4}$, and consider the particular geometry $K = x^{7/4}$. The form of expression (2.2) for $\bar{u}$ is maintained in region II if we take

$$f'(\xi) = 4\left(\frac{8\xi + H}{g_0}\right)^{1/2}. \tag{4.12a}$$

On comparing the expressions for $\bar{v}$ in I and II we find that $l(x) = Lx^{-3/4}$ to enable the similarity solution to exist. We have

$$\bar{v} = x^{-3/4}\left(\frac{7g_0 f'^3}{1536} - \frac{3Hf'}{32} + L\right). \tag{4.12b}$$

Finally, in (4.10) we replace $m(x)$ by $Mx^{-3/4}$ and rearrange to obtain

$$2|V_0|^2 + 1 = L(8\xi + H) - 7(8\xi + H)^{5/2}/48g_0^{1/2} + Mg_0^{1/2}(8\xi + H)^{1/2}. \tag{4.13}$$

We note that this equation only gives the magnitude of V_0. The phase may be found by writing $V_0 = |V_0| \exp i\phi$ in the $O(E\varepsilon^{-1})$ terms of equation (2.1d), which then reduces to

$$\phi_{yy}|V_0| + 2\phi_y|V_0|_y = 0.$$

So, once $|V_0|$ has been found from equation (4.13), the phase may be calculated.

We have now removed x from the problem. To solve for the disturbance velocity field, we simply start at the wall with $f(0) = f'(0) = 0$, $f''(0) = C$, a constant to be determined, and integrate (2.2) forwards until

$$f'f'' = 64/g_0. \tag{4.15}$$

This is the boundary between I and II, as shown in Fig. 3. In region II, equation (4.14) controls the flow, with $|V_0|^2 = 0$ at the boundaries. We match f, f' and f'' at ξ_L to find the constants H, L and M. We solve $|V_0|^2 = 0$ in (4.14) to find the boundary at ξ_U and calculate f, f' and f'' there. The mean flow equation (2.2) resumes control and is integrated to infinity. The boundary condition there requires $f' = 0$. We use a shooting method on C to correctly obtain this condition.

From the form of the numerical solution for large g_0, it was found that for $g_0 \gg g_{0_c}$ an asymptotic solution was readily available. If, in equations (4.12), (4.13), and (2.3a) we make the substitutions $\tilde{\xi} = g_0^{-1/5}\xi$, $F(\tilde{\xi}) = g_0^{1/5}f(\xi)$, $H = \tilde{H}g_0^{1/5}$, $L = \tilde{L}g_0^{-1/5}$ and $M = \tilde{M}g_0^{-3/5}$, the resulting system of equations contains no reference to the Görtler number at all. This is only possible because the boundary conditions for wall-jet flow, (2.3b), also enable the Görtler parameter to be scaled out. This is not the case for the Blasius boundary layer, where the condition $f'(\infty) = 1$ prevents this scaling. Further numerical work showed that whichever value of g_0 was used, the same scaled solution resulted, even for $g_0 < g_{0_c}$. The solution must therefore be considered in the following manner. The critical value of g_0 is taken from the linear theory for high wavenumbers. For immediately higher values of g_0, we may use weakly non-linear theory to construct the curves ξ_L and ξ_U between which the vortex activity is constrained. These curves merge into the full non-linear computed solution at some value of $g_0 > g_{0_c}$ after which the scaled solution is valid.

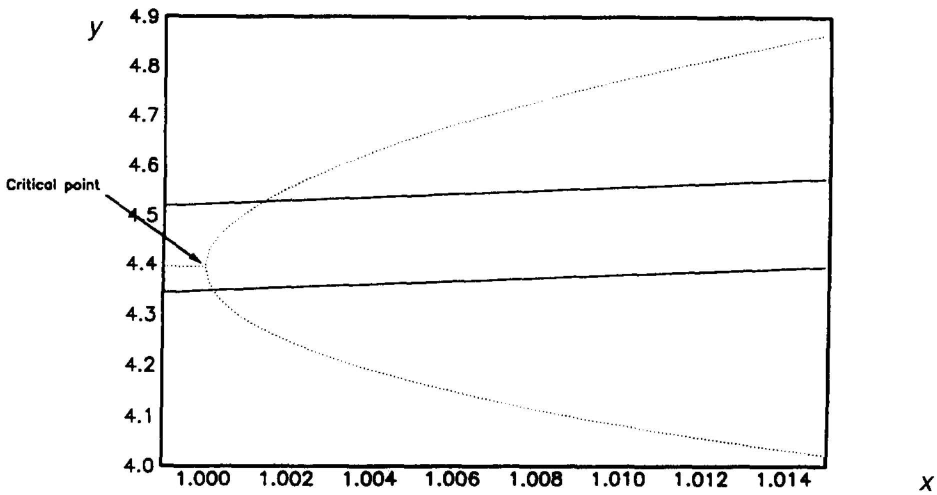

Fig. 4. The vortex boundaries given by weakly nonlinear and fully nonlinear theories, close to the neutral point Solid line: computed solution. Dotted line: weakly nonlinear asymptotic solution.

The core boundaries close to the critical point (x_c, y_c) where the vortex is starting may be found by using the weakly non-linear theory developed by HL. For the sake of brevity, we only state the result here; the full working for the wall-jet case may be found in Wadey [17]. For the particular case where the curvature function is $K = x^{7/4}$, the neutral point is $x_N = 1.0$ and the local Görtler number and wavenumber are $G_L = 6.34$ and $\varepsilon_L^{-1} = 4.0$ respectively, the general equation of the edges of the initial downstream vortex structure becomes

$$y_{U,L} = 4.395x^{3/4} \pm 3.45(x - 1.0)^{1/2} .$$

The values of the parameters chosen here correspond to a flow condition which is neutrally stable according to linear theory. This is shown in Fig. 4 together with the computed solution.

§5. Final remarks

The aim of this study has been to present some results on Görtler vortices in wall-jet flow, using the asymptotic theory developed by Hall [9, 13] and Hall and Lakin [15] for the Blasius boundary layer. The leading term in the expansion of the local Görtler number at high wavenumber is much higher for the wall jet on walls of either curvature than it is for the Blasius boundary layer, indicating that other things being equal, the Görtler instability becomes of practical importance in wall-jet flow only at much higher flow speeds or on more rapidly curved surfaces compared with the Blasius layer. In Figs 5 and 6 we show the velocity fields for Görtler numbers 1700 and 2700, together with the velocity profile in the absence of a vortex. In Fig. 5 we show how the effect of increasing the Görtler number beyond its critical value is to reduce the maximum downstream velocity and to move the position where $\bar{u}$ attains its maximum away from the wall. The presence of vortices has the same effect on the normal-to-wall velocity, $\bar{v}$. Figure 6 shows the size and position of the vortex activity for the two values of $g = G_L\varepsilon^4$ given above. The increase of 59% in the value of g in the two

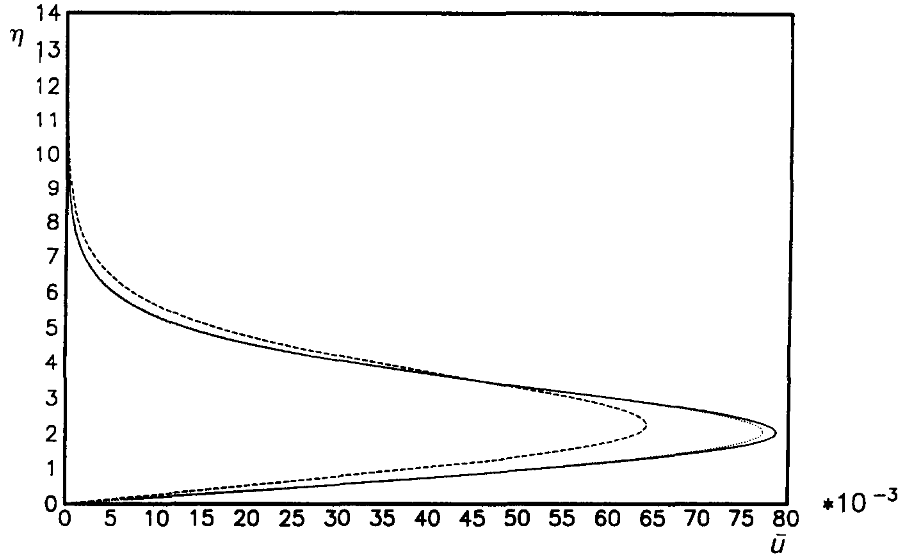

Fig. 5. The $\bar{u}$ velocity field. Solid line: velocity profile in absence of vortices. Dotted line: velocity profile for $g = 1700$. Broken line: velocity profile for $g = 2700$.

calculations whose results are presented in the figures only produces a 12% increase in $|V_0|^2$, although the proportion of the disturbance size increases by 50% from 0.04 to 0.06. It is interesting to note that the curve for $g = 1700$ does not lie within that for $g = 2700$ in Fig. 6, unlike the corresponding case in HL. This is a result of the different structure to the problem from that case. In HL, the upper and lower boundaries of the vortex activity, ξ_L and ξ_U, varied as g_0^{-3} and g_0 respectively for large g_0, whereas in the wall-jet case, both boundaries vary as $g_0^{1/5}$.

The removal of g_0 from the nonlinear problem considered in §4 is not restricted to the similarity situation. The Görtler number may also be scaled out of the general system of

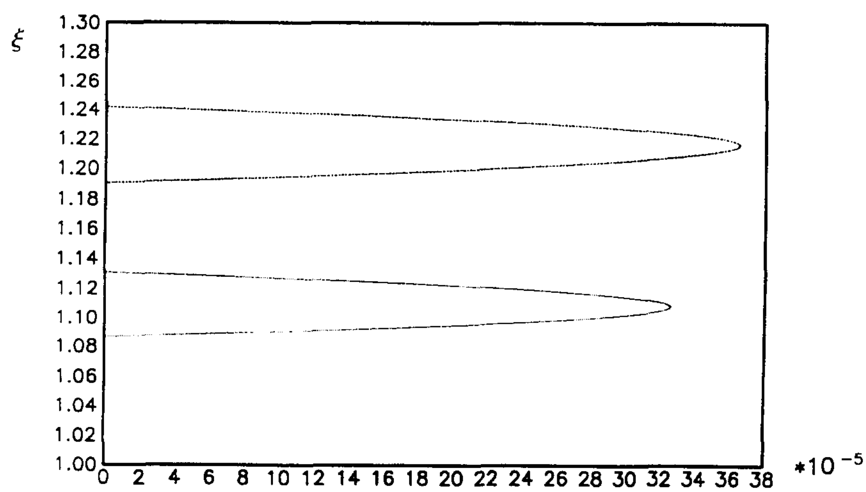

Fig. 6. Variation in size and position of the amplitude of the disturbance, $|V_0|^2$. Dotted line: velocity profile for $g = 1700$. Broken line: velocity profile for $g = 2700$.

equations, (4.12). The boundary-layer equations, (4.12a, b), may be made independent of g_0 via the transformation $u = \frac{1}{4} g_0^{-2/5} x^{-1/2} F'(\xi^{\dagger})$, where $\xi^{\dagger} = \frac{1}{4} y g_0^{-1/5} x^{-3/4} = g_0^{-1/5}\xi$. The expression for v is then found from the continuity equation as

$$\bar{v} = \tfrac{1}{4} g_0^{-1/5} x^{-3/4}(3\xi^{\dagger}F' - F)\,,$$

while the differential equation satisfied by F is found to take the same form as in (2.3a), with the same boundary conditions. The other boundary conditions in (4.11) have g_0 removed by use of the substitutions $h(x) = g_0^{1/5} h^{\dagger}(x)$, $l(x) = g_0^{-1/5} l^{\dagger}(x)$, and $m(x) = g_0^{-3/5} m^{\dagger}(x)$. We then have

$$u^{\dagger}(x, y^{\dagger}) = \left(\frac{2y^{\dagger} + h^{\dagger}}{K}\right)^{1/2}$$

and

$$v^{\dagger}(x, y^{\dagger}) = \frac{1}{6}\left(\frac{2y^{\dagger} + h^{\dagger}}{K}\right)^{3/2} K' - \frac{1}{2}\frac{\mathrm{d}h^{\dagger}}{\mathrm{d}x}\left(\frac{2y^{\dagger} + h^{\dagger}}{K}\right)^{1/2} + l^{\dagger}\,,$$

where $u = g_0^{-2/5} u^{\dagger}$, and $v = g_0^{-1/5} v^{\dagger}$. We observe that since $\xi = g_0^{1/5}\xi^{\dagger}$, the effect of increasing the Görtler number is to force the layer of activity away from the wall, and eventually into the free stream. Equation (4.10a) shows that when g_0 is scaled out of the problem, $|V_0|^2$ remains $O(g_0^0)$, $= O(1)$. Thus, at large g_0, the mean flow $u = g_0^{-2/5} u^{\dagger}$ is an order of magnitude smaller than the disturbance it creates. The conclusions of HL for the Blasius boundary layer remain unaltered for wall-jet flow although the structure of the flow after the onset of Görtler vortex growth is quite different. Of course, our inability to find solutions to the system of equations and boundary conditions, (4.11), other than that which is presented here does not mean that no such solutions exist. However, extensive numerical checking failed to produce any other solutions.

It is well known that in the Blasius case, vortices conserve their wavenumber as they progress downstream. Thus, by virtue of (3.18a), we note that if the same is true of Görtler vortices in wall jet flow, then the high wavenumber regime will utlimately be relevant to flows which occur in practice.

Acknowledgements

The author would like to thank Professor Philip Hall for his guidance throughout this work, and the Science and Engineering Research Council for providing financial assistance.

References

1. M.B. Glauert, The wall jet. *J. Fluid Mech.* 1 (1956) 625–643.
2. P.W. Carpenter, D. Bridson and P.N. Green, Features of discrete tones generated by jet flows over Coanda surfaces. *AIAA paper* (1986) 86–1865.
3. H. Görtler, On the three dimensional instability of laminar boundary-layers on concave walls. *Tech. Memo. Natn. Advis. Cttee. Aeronaut., Washington* (1940) No. 1375.

4. G. Hämmerlin, Über das Eigenwertproblem der dreidimensionalen Instabilität laminarer Grenzchichten an konkaven Wänden. *J. Rat. Mech. Anal.* 4 (1955) 279–321.
5. G. Hämmerlin, Zur Theorie dreidimensionalen Instabilität laminarer Grenzschten. *Z. angew. Math. Phys.* 1 (1956) 156–167.
6. Th. Herbert, On the stability of the boundary-layer on a curved wall. *Archs. Mech.* 28 (1976) 1039–1055.
7. J.M. Floryan and W.S. Saric, Stability of Görtler vortices in boundary-layers. *AIAA paper* (1979) 79–1497.
8. J.M. Floryan, Görtler instability of boundary-layers over concave and convex walls. *Phys. Fluids.* 29 (1986) 2380–2387.
9. P. Hall, Taylor–Görtler vortices in fully-developed and boundary-layer flows: linear theory. *J. Fluid Mech.* 124 (1982) 475–494.
10. Lord Rayleigh, On the dynamics of revolving fluids. *Scientific Papers* 6 (1916) 447–453.
11. J.T. Stuart, On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part one: The basic behaviour in plane poiseuille flow. *J. Fluid Mech.* 9 (1960) 352–370.
12. J. Watson, On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part two: the development of the solution for plane poiseuille flow and for plane Couette flow. *J. Fluid Mech.* 9 (1960) 371–389.
13. P. Hall, On the non-linear evolution of Görtler vortices in growing boundary-layers. *IMA J. Appl. Maths.* 29 (1982) 173–196.
14. P. Hall, The linear development of Görtler vortices in growing boundary-layers. *J. Fluid Mech.* 130 (1983) 41–58.
15. P. Hall and W.D. Lakin, The fully non-linear development of Görtler vortices in growing boundary-layers. *Proc. Roy. Soc.* A 415 (1988) 421–444.
16. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions.* Frankfurt: National Bureau of Standards (1964).
17. P.D. Wadey, Görtler Vortex Instabilities of Incompressible and Compressible Boundary Layers, Ph.D. Thesis, University of Exeter (1990).